The Constant Elasticity of Substitution Function

Dominique van der Mensbrugghe

Center for Global Trade Analysis (GTAP)
Department of Agricultural Economics
Purdue University

Table of Contents

The constant elasticity of substitution function

- Work horse specification in economics
  - Capital/labor substitution
  - Goods differentiated by region of origin (Armington)
- Special cases
  - Leontief (fixed proportions)
  - Cobb-Douglas (fixed value shares)
- Extensions/alternatives
  - CRESH (differentiated elasticity across pairs)
  - Additive CES (volumes and values add up)

The CES in detail

The primal objective function and constraint:

$$Cost = \min_{X_i} \left\{ \sum_i P_i X_i \right\}$$

subject to the constraint:

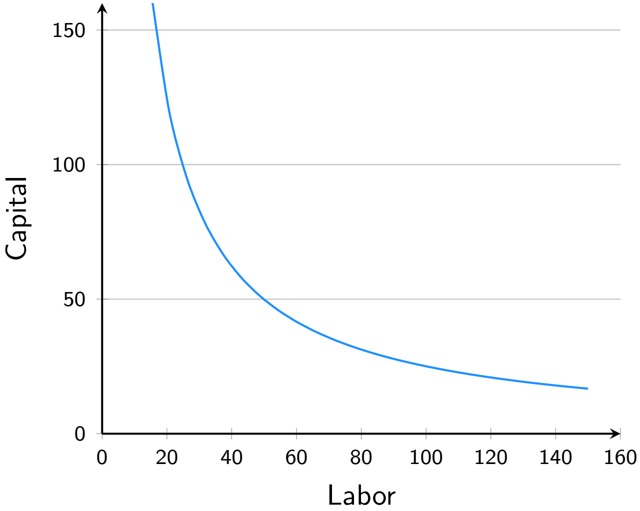
$$V = A \left[\sum_i a_i \left(\lambda_i X_i \right)^{\rho} \right]^{1/\rho}$$

$$U = \max_{X_i} \left\{ A \left[\sum_i a_i \left(\lambda_i X_i \right)^{\rho} \right]^{1/\rho} \right\}$$

subject to the constraint:

$$Income = \sum_i P_i X_i$$

CES curve: K vs. L



The CES in detail

The solution:

$$X_i = \alpha_i \left(A\lambda_i\right)^{\sigma-1} \left(\frac{P}{P_i}\right)^{\sigma} V \qquad \text{or } X_i = \alpha_i \left(\frac{P}{P_i}\right)^{\sigma} V$$

$$P = \left[\sum_i \alpha_i \left(\frac{P_i}{\lambda_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)} \iff P \cdot V = \sum_i P_i X_i$$

where we have made the following substitutions:

$$\sigma = \frac{1}{1-\rho} \geq 0 \Leftrightarrow \rho = \frac{\sigma-1}{\sigma}$$

and

$$\alpha_i = a_i^{1/(1-\rho)} = a_i^{\sigma} \Leftrightarrow a_i = \alpha_i^{1/\sigma}$$

Useful formulas

Cost or budget shares:

$$s_i = \frac{P_i X_i}{P \cdot V} = \alpha_i \left(A\lambda_i\right)^{\sigma-1} \left(\frac{P}{P_i}\right)^{\sigma-1} \Rightarrow s_i = \alpha_i \left(\frac{P}{P_i}\right)^{\sigma-1}$$

Input ratios:

$$\frac{X_i}{X_j} = \frac{\alpha_i}{\alpha_j} \left(\frac{P_i}{P_j}\right)^{-\sigma} \qquad \text{Armington example: } \frac{D}{M} = \frac{\alpha^d}{\alpha^m} \left(\frac{PD}{PM}\right)^{-\sigma}$$

Elasticity of substitution:

$$\frac{\partial \left(\frac{X_i}{X_j}\right)}{\partial \left(\frac{P_i}{P_j}\right)} \frac{\left(\frac{P_i}{P_j}\right)}{\left(\frac{X_i}{X_j}\right)} = -\sigma$$

Special cases

Leontief (fixed volume shares: $\sigma = 0$, $\rho = -\infty$)

$$X_i = \frac{\alpha_i}{\lambda_i}\frac{V}{A} \qquad P = \frac{1}{A}\sum_i \alpha_i \frac{P_i}{\lambda_i}$$

Cobb-Douglas (fixed value shares: $\sigma = 1$, $\rho = 0$)

$$V = A\prod_i (\lambda_i X_i)^{\alpha_i} \Rightarrow X_i = \alpha_i \frac{P}{P_i} V \qquad P = \frac{1}{A}\prod_i \left(\frac{P_i}{\alpha_i \lambda_i}\right)^{\alpha_i}$$

Perfect elasticity (homogeneous goods: $\sigma = \infty$, $\rho = 1$)

$$X = \sum_i \lambda_i X_i \qquad \frac{P_i}{\lambda_i} = P$$

Calibration

With known substitution elasticity and initialized values for the variables (including technology), the (dual) share parameters can be calculated using:

$$\alpha_i = \frac{X_i}{V}\left(\frac{P_i}{P}\right)^{\sigma} \iff \alpha_i = \frac{X_i}{V} \text{ if } P_i = P \, \forall \, i$$

Example:

Table: Armington calibration example: $\sigma = 3$

Value	Price	Volume	Dual share	Price	Volume	Dual Share
65	1	65	0.6500	1	65	0.6500
35	1	35	0.3500	1.25	28	0.5469
100	1	100		1	100	

Log-linearization

GEMPACK tradition

$$\dot{x}_i = \dot{v} + \sigma\left(\dot{p} - \dot{p}_i\right) + (\sigma - 1)\left(\dot{a} - \dot{\lambda}_i\right)$$

$$\dot{p} = -\dot{a} + \sum_i s_i \dot{p}_i - \sum_i s_i \dot{\lambda}_i$$

Advantages:

- Easy interpretation
- No need to calibrate, however need to add code to 'update' shares at each iteration

Technology/preference changes: twists

Change the ratio of two CES components assuming cost-neutrality:

$$R_{t+1} = \frac{X_{t+1}^1}{X_{t+1}^2} = (1+\tau)\frac{X_t^1}{X_t^2} = (1+\tau)R_t$$

$$C_{t+1} = P_t^1 X_{t+1}^1 + P_t^2 X_{t+1}^2 = P_t^1 X_t^1 + P_t^2 X_t^2 = C_t$$

τ is the so-called 'twist' parameter. Two examples:
Capital/labor:

$$R_{t+1} = \frac{K_{t+1}}{L_{t+1}} = (1+\tau)\frac{K_t}{L_t} = (1+\tau)R_t$$

Armington:

$$R_{t+1} = \frac{M_{t+1}}{D_{t+1}} = (1+\tau)\frac{M_t}{D_t} = (1+\tau)R_t$$

Technology/preference changes: twists

Solution:

$$\pi^1_{t+1} = \left[\frac{1 + s^1_t \tau}{1 + \tau}\right]^{1/(1-\sigma)} \Rightarrow \lambda^1_{t+1} = \pi^1_{t+1}\lambda^1_t$$

$$\pi^2_{t+1} = \left[1 + s^1_t \tau\right]^{1/(1-\sigma)} \Rightarrow \lambda^2_{t+1} = \pi^2_{t+1}\lambda^2_t$$

Armington example:

Table: Armington twist example: twist=10%, $\sigma = 2$

	Domestic	Import	M/D Ratio
Initial	80.0	20.0	0.250
After twist	78.4	21.6	0.275
Percent change	-2.0	7.8	10.0
Preference change	-0.02	0.08	

CET setup

CET is analogous to the CES—but used for allocating supply. Examples:

- Allocating land across uses
- Allocating export across destination regions
- Allocation multi-product production across uses (make)

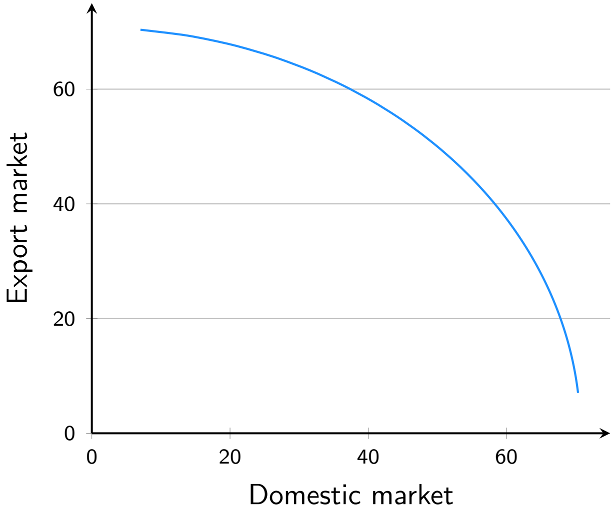
The primal objective function and constraint:

$$Revenue = \max_{X_i} \left\{ \sum_i P_i X_i \right\}$$

subject to the constraint:

$$V = A \left[\sum_i g_i \left(\lambda_i X_i \right)^{\nu} \right]^{1/\nu}$$

CET curve: D vs. E



CET solution

The solution:

$$X_i = \gamma_i \left(A\lambda_i\right)^{-1-\omega} \left(\frac{P_i}{P}\right)^{\omega} V \qquad \text{or } X_i = \gamma_i \left(\frac{P_i}{P}\right)^{\omega} V$$

$$P = \left[\sum_i \gamma_i \left(\frac{P_i}{\lambda_i}\right)^{1+\omega}\right]^{1/(1+\omega)} \iff P \cdot V = \sum_i P_i X_i$$

where we have the following substitutions:

$$\nu = \frac{1+\omega}{\omega} \iff \omega = \frac{1}{\nu - 1} \geq 0$$

$$\gamma_i = g_i^{-\omega} \iff g_i = \left(\frac{1}{\gamma_i}\right)^{1/\omega}$$

Additivity

The CES/CET preserve value additivity:

$$P \cdot V = \sum_i P_i X_i$$

but do not preserve volume additivity:

$$V \neq \sum_i X_i$$

This can be an issue, for example land supply. You can start with an initial total supply of 100 ha., but after a shock, you may end up with a total supply of more than 100 ha. or less.

ACES/ACET

One alternative is the additive forms of the CES/CET known as ACES/ACET (see also logit function). The starting point is a different objective function—a utility function that aggregates values and not volumes.

$$U = \min_{X_i} \left\{ \left[\sum_i a_i \left(\lambda_i P_i X_i \right)^{\rho} \right]^{1/\rho} \right\}$$

subject to the constraint:

$$V = \sum_i X_i$$

$$U = \max_{X_i} \left\{ \left[\sum_i g_i \left(\lambda_i P_i X_i \right)^{\nu} \right]^{1/\nu} \right\}$$

subject to the constraint:

$$V = \sum_i X_i$$

ACES/ACET Solution

ACES	ACET
$X_i = \alpha_i \left(\frac{P^c}{\lambda_i P_i} \right)^{\sigma} V$	$X_i = \gamma_i \left(\frac{\lambda_i P_i}{P^c} \right)^{\omega}$
$P^c = \left[\sum_i \alpha_i \left(\lambda_i P_i \right)^{-\sigma} \right]^{-1/\sigma}$	$P^c = \left[\sum_i \gamma_i \left(\lambda_i P_i \right)^{\omega} \right]^{1/\omega}$
$P \cdot V = \sum_i P_i X_i$	$P \cdot V = \sum_i P_i X_i$

N.B. Almost same calibration formula—however, must choose value of P^c, for example set $P^c = P$.

ACES/ACET log-linearized

ACES	ACET
$\dot{x}_i = \dot{V} + \sigma\left(\dot{p^c} - \dot{p}_i - \dot{\lambda^i}\right)$	$\dot{x}_i = \dot{V} + \omega\left(\dot{p}_i + \dot{\lambda^i} - \dot{p^c}\right)$
$\dot{p^c} = \sum_i r_i\left(\dot{p}_i + \dot{\lambda^i}\right)$	$\dot{p^c} = \sum_i r_i\left(\dot{p}_i + \dot{\lambda^i}\right)$

where $r_i = X_i/V$. The price/volume split matters for this specification. In the standard CES/CET specification only budget shares matter, and price/volume levels are irrelevant. This implies that initialization/calibration matters.

ACES/ACET twists

ACES	ACET
$\pi^1_{t+1} = \left[\dfrac{1 + r^1_t \tau}{1 + \tau}\right]^{1/\sigma}$	$\pi^1_{t+1} = \left[\dfrac{1 + \tau}{1 + r^1_t \tau}\right]^{1/\omega}$
$\pi^2_{t+1} = \left[1 + r^1_t \tau\right]^{1/\sigma}$	$\pi^2_{t+1} = \left[\dfrac{1}{1 + r^1_t \tau}\right]^{1/\omega}$

Twist parameters depend on the volume shares, not the value shares. Note that the ACES/ACET twists are calibrated to neutrality of the composite price. This does not imply neutrality of the average price or of utility.