Structural Parameters of Trade Models with Firm Heterogeneity

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ABSTRACT

While various firm heterogeneity models of trade have recently emerged in the CGE literature, their mainstream adoption in trade policy analysis has been limited partly due to lack of available parameter estimates at the disaggregate sector level. The firm heterogeneity theory in quantitative trade models assume firm productivity as well as firm size distributions are characterized by power-laws, especially in the upper tail of the distribution where rare events such as exporting occurs. In recent empirical studies, it is reported that power-laws may not be suitable for the entire range of the distribution. Therefore, it is paramount to identify a minimum threshold above which the power-law provides a good fit for the data. This is especially important in estimating the structural parameters of the firm heterogeneity model for use in policy analysis as biased estimates may distort trade volumes and welfare responses. In this paper, we use maximum-likelihood and Kolmogorov-Smirnov (KS) statistic to estimate the minimum threshold for truncating the data which then allows us to estimate the shape parameter of firm size and productivity distributions based on power-law and ORBIS firm-level data. We then impute the elasticity of substitution across varieties that are appropriate to use in firm heterogeneity models of trade. We provide parameter values for productivity distribution and substitution elasticity for 19 GTAP manufacturing sectors.

Keywords: Melitz, Power-law, Pareto Distribution, Productivity, Firm Heterogeneity, Shape Parameter, Elasticity

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1. Introduction

Since the seminal work of Melitz (2003), a growing literature on firm heterogeneity models has emerged with the ability to explain trade and welfare effects of economic integration in greater precision. One of the challenges in advancing their widespread adoption for informing policy analysis is the lack of information on structural parameters. Chosen parameter values play a critical role in determining trade volumes and welfare responses to policy changes in trade models. The impact of parameters is more pronounced in firm heterogeneity models since there are two key parameters to consider: the shape parameter of productivity distribution (γ) and the elasticity of substitution across varieties (σ).

While there are a number of methodologies prevalent in the current literature for obtaining these parameters, they often lack consistency with the underlying firm heterogeneity theory, indicating a clear need for continued efforts towards theory-consistent parameterization of firm heterogeneity models (Ahmad and Akgul, 2018). To simplify the complex task of identification, many firm heterogeneity models adopt parameter values from the existing literature (Balistreri, Hillberry, and Rutherford, 2011; Eaton, Kortum, and Kramarz, 2011; di Giovanni and Levchenko, 2013; Melitz and Redding, 2013). A potential disadvantage of this approach is that shape parameters and elasticities are often estimated based on different methodologies which may not be consistent with firm heterogeneity. For example, elasticity values are estimated using traditional gravity models with Armington assumption of national product differentiation which is inconsistent with the underlying Dixit-Stiglitz love-of-variety structure.

In this paper, we propose a simple method to estimate the structural parameters of firm heterogeneity models that relies on the theoretical relationship between the size distributions of firms and the γ and σ parameters, extending the work of Ahmad and Akgul (2018). In firm heterogeneity models, the common assumption is that firm productivity follows the Pareto distribution, which is a power-law model with an exponent γ, i.e. the shape parameter of productivity distribution. In general, when firm productivity is from a Pareto distribution with shape parameter γ, then the firm size also follows a Pareto distribution, however, with a different shape parameter, α, which in fact is the ratio of the two structural parameters: $\alpha = \frac{\gamma}{\sigma-1}$ (di Giovanni and Levchenko, 2013). Since α can be estimated directly from firm-level data, it provides a useful way to infer the ratio of structural parameters in the firm heterogeneity model. An expanding body of empirical studies uses this property to consistently estimate α. However, since this expression is a combination of γ and σ, it is not possible to estimate the individual structural parameters in these studies. Therefore, more information is needed for separate identification of both these parameters.

One approach is to use existing estimates of elasticity of substitution (Broda and Weinstein, 2006) and then impute the shape parameter (Chaney, 2008). While this method circumvents some of the difficulties associated with parameter identification, it has two drawbacks: (i) Estimates for elasticity of substitution are often obtained from traditional gravity equations that depend on the Armington assumption, which is fundamentally inconsistent with firm heterogeneity theory and reflects only the demand-side heterogeneity in the model (ii) The resulting values for the shape parameter typically are not sector and region-specific; therefore, they do not capture the significant variation of heterogeneity along these dimensions. Not accounting for these drawbacks is likely to lead to biased estimates of parameters in the calibrated model.

To overcome these methodological issues, we propose a simple methodology to obtain structural parameters of firm heterogeneity using the actual distribution of firm productivity to obtain the estimates of the shape parameter directly from the data assuming power-law model. Using firm size and productivity distribution together makes a potentially useful tool for empirical research on estimating the structural parameters of firm heterogeneity, since it can be used on the same firm-level dataset and without needing a number of model-based equations.

We also address one of the criticisms in the literature of assuming a power-law model: the real world variables do not follow the power-law model over the entire range. In fact, data follows the power-law model only above a lower bound. In particular, the power-law function $p(x) = Cx^(-\alpha)$ diverges as x approaches zero, given a positive value of α. This means that there is a minimum lower-bound x_{min} below which the distribution deviates from the power-law. That is why we focus on the tail of the distribution rather than the entire range in firm heterogeneity.

In order to obtain an unbiased estimate of the shape parameter of productivity distribution, it is paramount to focus on the upper tail of the sample, where the power-law distribution applies which corresponds to the observations that are higher than the lower bound. This implies that in order to have an accurate estimation of the shape parameter, first we need to obtain an accurate estimate of the lower-bound. If the lower-bound value is too low, the sample does not follow a power-law and we are trying to fit a non-power law data to a power-law distribution which may likely provide a biased shape parameter. If the lower-bound value is too high, then we are omitting relevant information from the sample which may increase the statistical error on the parameter estimates as well as the bias from finite size effects (Clauset, Shalizi, and Newman, 2009). While both cases are not desirable, it may be advised to err on the higher minimum value as this will reduce the sample size. However, the error will be less severe compared to the case with higher sample size and a lower x_{min} (Clauset, Shalizi, and Newman, 2009).

A common method to obtain a lower bound is based on visualization. There are two ways to estimate visually. The first one is to plot the shape parameter as a function of the lower bound, i.e. truncation point, and identify where the value fluctuates and where it becomes stable. Then, choose the point where the relationship becomes stable. The second method is to depict the log-log plot and identify the point where the PDF or the CDF of the distribution becomes relatively straight.

In order to avoid the subjectivity of this visualization method, Clauset, Shalizi, and Newman (2009)) offer a more robust and methodical approach in choosing a lower bound. In this paper, we adopt their approach, which is based on minimizing the distance between the power law model and the empirical data. They suggest choosing the value of x_{min} that makes the model and the distribution of the empirical data as similar as possible. To quantify the distance between two distributions, they use the Kolmogorov-Smirnov (KS) statistic, which is defined as the maximum distance between the CDFs of the data and the fitted model.

In this paper, we combine the methodology in Ahmad and Akgul (2018) and Clauset, Shalizi, and Newman (2009) to estimate the structural parameters of firm heterogeneity, namely the shape parameter of productivity distribution and the elasticity of substitution across varieties. We analyze the power-law model in three steps following Clauset, Shalizi, and Newman (2009). In the fourth step we impute the value of elasticity of substitution across varieties based on the power-law model. In a broad outline, our methodology includes the following steps to identify the structural parameters of firm heterogeneity:

1) We estimate the lower bound parameter x_{min} by minimizing the distance between the empirical distribution and model distribution based on the Kolmogorov-Smirnov (KS) statistic.
2) Based on the value of x_{min} we estimate the power-law exponent of firm size and firm productivity using the method of Maximum Likelihood.
3) We compare the power-law model with alternative distributions such as the exponential and log-normal distributions using a Likelihood Ratio test.
4) We use the estimates of power-law exponents in firm size and firm productivity to impute the elasticity of substitution across varieties in the heterogeneous sector.

We illustrate this methodology by using firm-level ORBIS database and focusing on the EU. We select the EU manufacturing sectors as there is considerable heterogeneity across firms and differentiation across the available varieties, which can better reflect the characteristics of the underlying Melitz theory. We choose GTAP Version 10 sectoral classification where 19 manufacturing sectors are identified. For firm size, we use two variables: firm operating revenue and number of employees. In order to calculate the firm productivity levels, we use a common labor productivity metric, where we divide the operating revenue to the number of employees. Alternatively we use an index total factor productivity (index TFP) measure as firm productivity proxy.

The remainder of this paper is organized as follows. We begin our appraisal in Section 2 with a summary of the empirical methodology. In Section 3 we discuss the data. In Section 4 we move on to the estimation results. Section 5 concludes the paper.

2. Empirical Methodology

2.1 Firm Size Distribution in a Melitz World

A stylized fact in the empirical trade literature with heterogeneous firms is that the tail behavior of the firm size distribution follows a Power Law, specifically Pareto distribution (Axtell, 2001). In fact, it is argued that the tail behavior is well approximated by a Zipf Law, where full granularity is realized with the power law exponent near unity.

The probability density function (PDF) of a continuous power-law model, $p(x)$, can be described by a non-negative random variable X, i.e. the observed variable,

$$p(x)dx = \Pr\left[x \leq X < x + dx\right] = Cx^{-\alpha}dx \tag{1}$$

where $C > 0$ is a constant, x is the data that are modeled by the distribution and α is the power-law exponent, i.e. the shape parameter of the distribution.[1]

As discussed in Clauset, Shalizi, and Newman (2009), Equation (1) does not hold for all x. In fact, it may diverge as x approaches zero. Therefore, the power-law model applies only above a lower bound, which is denoted as x_{min}. The resulting PDF of a continuous power-law model is given by

$$p(x) = \frac{\alpha - 1}{x_{min}}\left(\frac{x}{x_{min}}\right)^{-\alpha} \tag{3}$$

where x_{min} is the lower bound for the power-law model, data follows a power-law for $x \geq x_{min}$ and α is the corresponding power-law exponent. The associated complementary cumulative distribution function (CCDF, i.e. 1-CDF) is given by

$$p(x)dx = \Pr\left[X \geq x\right] = \int_{x}^{\infty} p(x)dx = \left(\frac{x}{x_{min}}\right)^{-\alpha+1} \tag{4}$$

In the heterogeneous firms model, if the firm productivity distribution can be described by a power-law, then the firm size also follows the same distribution with a different value for the power-law exponent. As a result, when firm productivity (φ) follows Pareto distribution with $\varphi \sim Pareto(x_{min}, \gamma)$, the firm size ($S_i$) distribution can also be described by the Pareto distribution with $S_i \sim Pareto(x_{min}^{1-\sigma}A, \frac{\gamma}{\sigma-1})$.[2]

We analyze the power-law model in three steps following Clauset, Shalizi, and Newman (2009). For the implementation of this methodology, we rely on a power-law fitting library in Python that was developed by Alstott, Bullmore, and Plenz (2014). We now turn to the description of this methodology for each of these steps.

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[1] Pareto distribution is considered as a power-law since it satisfies

$$\Pr\left[X \geq x\right] = \left(\frac{x}{x_{min}}\right)^{-\alpha} \tag{2}$$

where $\frac{x}{x_{min}} > 0$ is the lower bound for the power-law model, which can be described by $C = x_{min}^{\alpha}$, and $\alpha > 0$ is the shape parameter of the Pareto distribution.If $\alpha = 1$, a special case of Pareto distribution is achieved which is known as the Zipf's Law.

[2] Please see Appendix A for the detailed derivation.

2.2 Estimating the Lower Bound Parameter

Following Clauset, Shalizi, and Newman (2009), a numerical method is used to select the x_{min} that yields the best power-law for the data. Specifically, for each x_{min} over some reasonable range, the Kolmogorov-Smirnov (KS) statistic is utilized to quantify the distance between the empirical distribution and model distribution. While other measures can also be used for quantifying distance, the KS statistic has been shown by Clauset, Shalizi, and Newman (2009) to perform well in these estimations.[3] The KS statistic is computed as the maximum distance between the CDFs of the data and the power-law model

$$D = max|S(x) - P(x)| \tag{5}$$

where $S(x)$ is the CDF of the data and $P(x)$ is the CDF for the theoretical power-law model, both for observations with value $x \geq x_{min}$. The estimated x_{min} is the one that provides the best fit to the data by minimizing the distance D.

2.3 Estimating the Power-Law Exponent

The first step of estimating the exponent in a power-law model requires the correct identification and estimation of the lower bound parameter, x_{min}. Once the value of x_{min} is estimated based on the methodology described in Section 2.2 above, the power-law exponent is estimated using the method of Maximum Likelihood.

The Log likelihood function $\mathcal{L}$ is given as

$$\mathcal{L} = \ln p(x|\alpha) = \ln \prod_{i=1}^{n} \frac{\alpha - 1}{x_{min}} \left(\frac{x}{x_{min}} \right)^{-\alpha} \tag{6}$$

We maximize $\mathcal{L}$ with respect to α such that $\frac{\partial \mathcal{L}}{\partial \alpha} = 0$. The resulting Maximum Likelihood Estimator is given by

$$\hat{\alpha} = 1 + n \left[\sum_{i=1}^{n} \ln \frac{x_i}{x_{min}} \right]^{-1} \tag{7}$$

where x_i with $i = 1, 2, ..., n$ are the observed values of x such that $x \geq x_{min}$.

[3] Alternative metrics such as Kuiper or Anderson-Darling are also used for quantifying distance in the literature to capture more sensitivity to differences at the tails. However, we prefer the KS statistic since there is not much evidence that Kuiper performs better, while Anderson-Darling reduces the size of the data considerably (Clauset, Shalizi, and Newman, 2009).

2.4 Likelihood Ratio Tests for Alternative Distributions

If the power-law is a good fit for the dataset, we should also investigate whether alternative distributions provide a better fit than the power-law. In order to make that evaluation we follow Clauset, Shalizi, and Newman (2009) and use the likelihood ratio test which computes the logarithm of the ratio of the likelihoods of the data between two distributions. We compare the power-law model to the exponential and log-normal distributions. A positive value of the likelihood ratio indicates that the power-law model is a better fit compared to the alternative, while a negative value indicates that the power-law model is a worse fit compared to the alternative.

2.5 Computing the Elasticity of Substitution

In the firm heterogeneity literature, when firm productivity follows a power-law model, in this case Pareto distribution, firm size also follows a power-law model, i.e. Pareto distribution. The power-law exponents for firm productivity and firm distributions are different. However, since they are theoretically connected, we can infer other information on structural parameters such as the value of elasticity of substitution across varieties by using these two power-law exponents.

More specifically, if firm productivity is from a power-law model with exponent γ, then the firm size also follows a power-law model with exponent $\alpha = \frac{\gamma}{\sigma - 1}$ (di Giovanni and Levchenko, 2013), where σ is the elasticity of substitution. Since α and γ can be estimated directly from firm-level data, this theoretical relationship can be used to infer σ.

3. Data

In this paper, we use the ORBIS database to obtain annual firm-level financial data on the manufacturing sector in the EU (excluding Italy). ORBIS uses both administrative and public data to provide firm-level information for over 200 million companies worldwide. Several procedures have been undertaken in ORBIS to verify the quality of reported data, including an indexation strategy to ensure the uniqueness of individual firms as well as an analysis to detect unusual variations of financial values between years. Detailed ownership information for firms is also provided in this database. In ORBIS firms are classified into industries based on the Statistical Classification of Economic Activities in the European Community, referred to as NACE. The current version of NACE classification is Revision 2 with four hierarchical levels. In ORBIS, level 4 of NACE is used where 615 classes are identified by four-digit numerical codes.

We only focus on manufacturing sectors and restrict the time frame of our study to the 2012-2016 period. While we consider an aggregated manufacturing sector, we also recognize that a more disaggregated sectoral analysis may result in different power-law exponents and lower bound values due to fewer observations, larger firms and relatively higher dispersion. For comparison, we conduct the estimation analysis for the pooled database with an aggregated manufacturing sector as well as disaggregated sectors.

We start off with the 4-digit NACE codes for sectoral identification. We, then map 4-digit NACE codes to GTAP Version 10 sectoral identification - GSC3 Sectors (Aguiar et al., 2019), where 65 sectors are identified in total with 19 manufacturing sectors. GTAP provides sectoral classification by reference to the ISIC Revision 4. Therefore, mapping is conducted in two steps: (i) 4-digit sectors in NACE Revision 2 (Rev.2) are mapped to ISIC Revision 4 classification, (ii) ISIC Revision 4 (Rev.4) sectors are mapped to GTAP GSC3 sectors.

3.1 Firm Size

Table 1 reports descriptive statistics for total revenue, number of employees and tangible fixed assets values for firms in the EU excluding Italy. There are 216.206 observations in our final database, where each observation corresponds to one firm in the NACE 4-digit sector with firm size information pooled across the available years (2012-2016).

There are two measures we rely on for firm size: (i) operating revenue (deflated by price index), (ii) number of employees. Similarly, for productivity we use two measures: (i) labor productivity, where operating revenue is divided by the number of employees at the firm and (ii) index measure of Total Factor Productivity (index TFP), where a Cobb-Douglass production function is used to calculate TFP. Results are provided based on all measures.

Table 1. Summary Statistics of EU Manufacturing Firms in Thousands USD (excluding Italy).

	Operating Revenue	**Number of Employees**	**Tangible Fixed Assets**
count	216,206	216,206	183,885
mean	15,586,317	42	4,078
std	300,187,424	369	354,005
min	290	1	-608
25%	149,433	2	20
50%	598,866	6	128
75%	3,055,939	21	701
max	63,453,347,840	104,067	150,701,472

Notes: Count indicates the number of observations in the database; mean indicates the mean value; std indicates the standard deviation; min indicates the minimum value; max indicates the maximum value; 25%, 50% and 75% indicate the percentiles of the distribution; values for operating revenue and tangible fixed assets are in thousands USD; numbers are reported for employees. Both operating revenue and tangible fixed assets are deflated by the price index.

Source: Author elaboration from ORBIS database.

The total number of observations for firm operating revenue and number of employees are 216,206. For tangible fixed assets we have a slightly smaller sample size of 183,885. Operating revenue and tangible fixed assets are deflated by the price index. We use the total revenue and the number of employees to analyze the firm size distribution. While the maximum total revenue available in the database is approximately 63,453 billion USD, the minimum value is reported as 290. Since values are in thousands of USD, 290 corresponds to a small firm with revenue \$290,000. The average operating revenue pooled across years is 15,586 million USD. Information on tangible fixed assets is used for measuring capital to calculate index TFP.

3.2 Firm Productivity

In this paper, we use two measures to calculate firm productivity. The first one is labor productivity (Y/L) where Y is total revenue.[4] Labor productivity is a popular measure of firm productivity as most firms provide information on revenues and amount of labor used. Therefore, there is adequate firm coverage for meaningful policy analysis.

Since labor productivity does not consider the intensity in use of excluded inputs such as capital, which may act as a substitute to labor in certain production processes, we also use a second measure for firm productivity based on firm's total factor productivity (TFP). Generally, TFP is calculated using either an index number approach (Solow residual technique) or estimation-based methods. Under the index number approach, TFP relates output to a weighted sum of inputs with the weights determined from aggregate (sector or economy-/wide) sources on labor and capital shares of income. Estimation based TFP's are the residuals from an estimated production function using firm-level (or plant-level) data.

The index TFP measure is based on an index number approach, first suggested by Solow (1957) to account for productivity growth due to technological progress. Despite its longevity, it still remains one of the more popular ways to determine TFP at both aggregate and sectoral levels (Del Gatto, Di Liberto, and Petraglia, 2011). Given a standard Cobb-Douglas production function, the index-number TFP for each firm is computed as: $y_{it} - \theta l_{it} + (1 - \theta)k_{it}$ where all variables are in natural log terms and θ is equal to the labor share of income. We assume the same value for labor share of income across sectors where $\theta = 0.51$.

4. Results

This section illustrates the results of power-law fits and parameter estimations. We estimate the lower bound on the power-law behavior and the power-law exponent in firm size as well as in productivity. As a proxy for firm size, we consider firm-level information on operating revenue and number of employees. As a proxy for productivity, we use a standard measure of labor productivity, the ratio of operating revenue to number of employees as well as an index measure, index TFP,

[4] Alternatively, to account for intermediate inputs, value added (revenue-cost of inputs) should be used as the measure of output. However, value added is not always available in a firm's financial data. For example, many countries, including the U.S., do not require firms to provide information on material and labor costs, leading to insufficient coverage of value added measures in the ORBIS database.

using a Cobb Douglas production function. We also use goodness of fit tests to compare power-law fit against alternative distributions such as the exponential and log-normal distributions. We then provide sensitivity analysis to observe the effect of lower-bound on the power-law exponent estimates. Finally, we report values for elasticity of substitution across varieties for 19 GTAP manufacturing sectors based on the estimated power-law exponents.

This section is divided into three parts. Section 4.1 presents the results for 19 GTAP manufacturing sectors. Section 4.2 discusses the structural parameters of firm heterogeneity with a focus on the elasticity of substitution. Section 4.3 conducts a sensitivity analysis to observe how the selection of lower bound affects power-law exponents.

4.1 Power-Law Exponents for GTAP Manufacturing Sectors

Figure 1 reports the ORBIS data for firm size and productivity in the EU (excluding Italy) using GTAP sectoral definition. Red points in the figure indicate observations that are higher than the optimal lower bound, $\frac{x}{min}$ values, while pink points indicate the observations that are lower. In other words, the beginning of red points represents the truncation level for each variable with the observations that are above x_{min} used in the power-law fit. Figure 1 shows that the optimal x_{min} value for power-law fit varies across sectors and alternative measures of firm size and productivity. In general, the truncation point of the data is high suggesting that the tail of the distribution is a better candidate for power-law fit.

For example, in sectors such as chm and eeq, the lower-bound for operating revenue is much higher than for otn and p_c. Based on optimal $\hat{x}_{min}$ values, top 35% of the data is used in the power-law fit for otn and p_c sectors, while only the top %1 chm and eeq sectors. A similar variation is observed for optimal lower bounds in number of employees for the majority of the sectors. Exceptions are tex, wap, lea and lum, where there is higher variation the optimal values for number of employees as opposed to operating revenue. The optimal $\hat{x}_{min}$ values are close for firm productivity, especially when labor productivity is used as a proxy.

Figure 2 shows the Maximum Likelihood estimates of the power-law exponents in firm size and productivity based on optimal $\hat{x}_{min}$ values reported earlier. There is a slight variation in the value of power-law exponents across GTAP sectors. Values are between 1.5 - 3.5, which satisfies the mathematical constraint $\gamma > \sigma - 1$. When firm size is measured by operating revenue, the distribution is slightly narrower with values between 1.5 - 2.6, while when firm size is measured by number of employees, the distribution is wider with values ranging between 1.5 - 3.4. Overall, sectors are found to be less heterogeneous with higher $\hat{\alpha}$ values when number of employees are used as firm size proxy.

Power-law exponents are higher for firm productivity compared to size clustering between 2 - 3, which suggests that productivity is less heterogeneous than size within the GTAP sectors. On the other hand, distribution of $\hat{\alpha}$ values across sectors is similar independent of the productivity proxy.

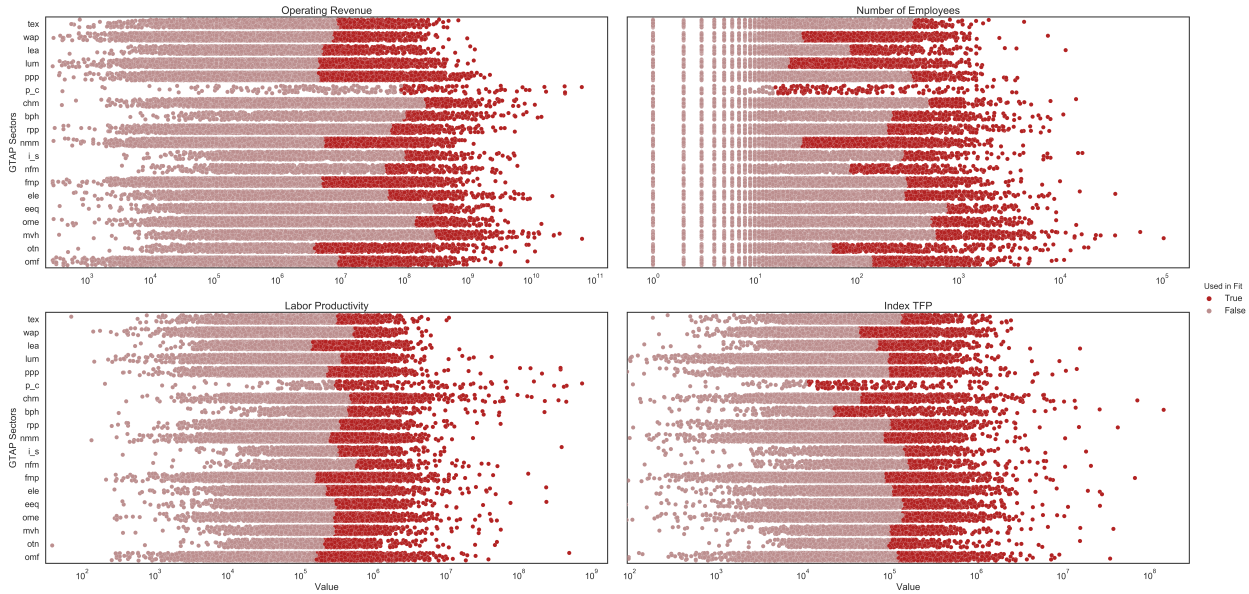

Figure 1. Data Distribution in the GTAP Manufacturing Sectors for Firm Size and Productivity (EU excluding Italy)

Notes: This figure reports the firm-level data in GTAP V.10 manufacturing sectors with optimal lower-bound for firm size and productivity. Red points show observations used in the power-law fit, while pink points show observations that are not used in the fit.)

Source: ORBIS database and author calculations.

Variation in $\hat{\alpha}$ implies that the average firm size and productivity to survive in each sector are different. For example, the average firm size to survive in the otn sector is larger compared to eeq with $\hat{\alpha}$ 1.5 for otn and 2.5 for eeq. For labor productivity, the power-law exponent is 1.5 in p_c, while it is 3.0 for wap. This result suggests that the average firm productivity for survival in the p_c sector is larger than that of wap. [5]

Empirical CCDF and power-law fits to our database for firm operating revenue in GTAP manufacturing sectors are shown in Figure 3. Complementary cumulative distribution function (CCDF; $1 - CDF$; $p(X_i \geq x)$) is reported based on the corresponding x_{min} values. For convenience, esti-

[5] As discussed in di Giovanni, Levchenko, and Rancire (2011), there may be a differentiation of power-law exponents between exporting and non-exporting firms based on the theory. They observed that exporters have a lower shape parameter in firm size compared to non-exporters indicating that the size distribution of exporters are systematically more fat-tailed than the size distribution of non-exporters. We isolated exporting and non-exporting firms in the EU and calculated the optimal x_{min} to estimate the MLE for power-law exponent of exporters and non-exporters individually. We observe that the size distribution of exporters is lower than that of non-exporters when operating revenue is used as firm size proxy; while it is higher when firm size is measured by number of employees. For firm productivity the distinction is more clear with the productivity distribution of exporters have systematically lower power-law exponents compared to non-exporters both when labor productivity and index TFP are used respectively as firm productivity proxies.

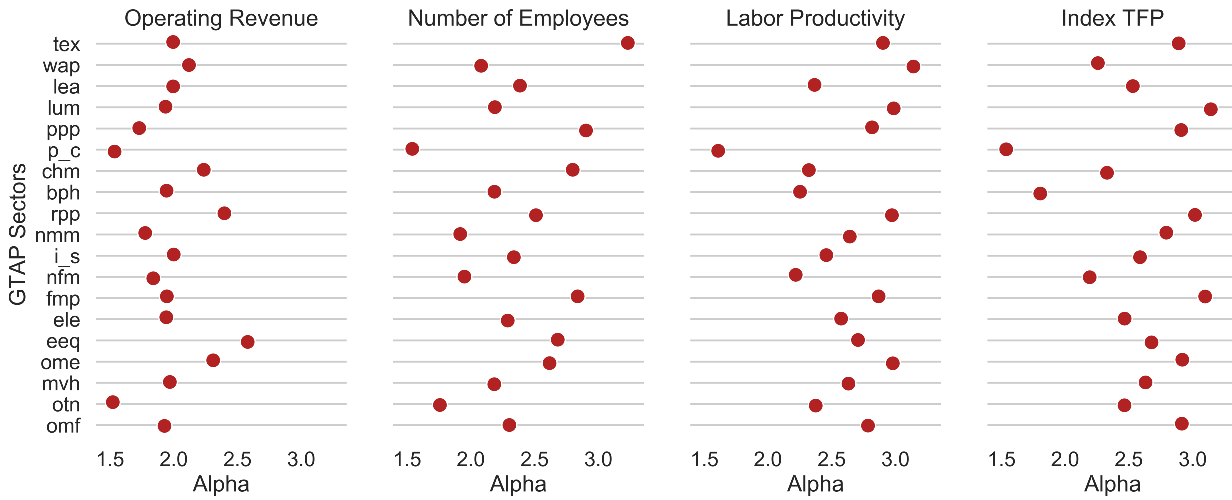

Figure 2. Estimates of the Power-law Exponent ($\hat{\alpha}$) in GTAP Manufacturing Sectors (EU excluding Italy)

Notes: Estimates of the Power-law exponent in GTAP manufacturing sectors of the EU excluding Italy (A) Operating Revenue, (B) Number of Employees, (C) Labor Productivity, and (D) Index Total Factor Productivity.

Source: Author calculations.

mates of power-law exponents are also reported on the figure. Results show that there is variation in terms of the fits across sectors. For several sectors, power-law provides a strong fit such as mvh, chm, bph and eeq. We also observe some sectors where empirical and power-law fits deviate at the tail such as in fmp, ppp, nmm, otn, omf and lum. The difference in model fit performance does not seem to be stemming from the power-law exponents. For example, while mvh and fmp sectors share a similar value for power-law exponent - 1.946 for fmp, 1.969 for mvh - model fit is much better in mvh compared to fmp.

Empirical CCDF and power-law fits to firm labor productivity in GTAP sectors are shown in Figure 4. For the majority of the GTAP sectors, power-law provides a good fit with slight deviation from the empirical CCDF such as in sectors tex, wap, lea. Only for a few sectors does the model fit diverge substantially such as in ppp, ome, eeq and chm.

The Empirical CCDF and power-law model fits of alternative measures of firm size and productivity of number of employees and index TFP are presented in Figure B.1 and Figure B.2 in Appendix B, respectively.

In light of the variation of model fit performance across sectors, the question remains: is power-law a good choice for firm size and productivity in GTAP sectors? It is reasonable to conclude that power-law is a good choice for specific sectors, while there is a possibility that other distributions may perform better for the rest as tail behavior in their model fit diverges substantially from the data.

To address the question of power-law performance in each GTAP sector, we test the fits against alternative distributions and report the log-likelihood ratio with corresponding p-values. Positive values of LR mean that the power-law model provides a better fit compared to the alternative distribution, while negative values mean that the power-law model provides a worse fit compared to the alternative distribution. Reported p-values indicate the significance of the test. If p-value is small, then the model with a worse fit may be rejected against the alternative. In this paper, we choose a p-value of 5% such that if the reported p-value is larger than 0.05, it is not possible to choose between the two models[6].

Figure 5 shows the LR test results for exponential distribution. Red points indicate the LR values are statistically significant at the 5% level; while pink points indicate insignificance. For all variables and all sectors, the LR values are positive indicating that the power-law model is a better fit compared to exponential. Majority of the values are statistically significant at the 5% level with the exception of tex and eeq for number of employees and i_s for labor productivity, which suggests that the test is inconclusive for those sectors.

Figure 6 shows the LR test results for log-normal distribution. In this case, the LR values are mostly negative; however, they are also not statistically significant. For operating revenue, all sectors with the exception of rpp result in negative LR values indicating that power-law model does not perform better against log-normal. However, the p-values for most of those sectors are high indicating that the test is inconclusive. The statistically significant values are encountered in sectors such as tex, ppp, nmm, fmp, otn and omf.

Similarly, negative LR values are found for most of the sectors when number of employees is used as firm size proxy, with only a few showing zero LR values. However, the LR values are only statistically significant for wap, lum, p_c, nmm and nfm, where we can conclude that the log-normal distribution performs better than the power-law model. There are more positive LR values encountered for labor productivity with only ele and ome sectors statistically significant. The majority of the sectors show negative LR values for index TFP with only wap and p_c sectors statistically significant. Overall, the test is not conclusive for most of the sectors across four variables on whether power-law provides a better fit than log-normal.

Based on the LR tests and the CCDF fits, we observe that while labor productivity may follow a power-law model, operating revenue may not in some GTAP sectors. For example, for operating revenue power-law performs worse than log-normal in sectors such as tex, ppp, nmm, fmp, otn and omf with negative LR values that are statistically significant. Figure 3 also demonstrate that power-law fits for those particular sectors do not perform well in the tail. On the other hand, power-law mostly performs well for labor productivity distribution in those sectors with better fits and LR values close to zero or positive against log-normal.

Hence we see that firm size and productivity may not follow the same distribution within the same

[6] Brzezinski (2014) uses 10% as the significance level. We observe similar results when 10% is used. For log-normal distribution, the LR values are statistically significant for the same sectors.

sector in our dataset. This indicates that there may be additional sources of heterogeneity besides productivity in some sectors. In fact, Amand and Pelgrin (2016) discuss that one-to-one relationship between exports data and productivity are weakened when there is additional heterogeneity. For example, market access costs could be heterogeneous in which case firm size such as exports may follow a different distribution compared to productivity because now market access costs are also informing export participation. There could be low-productivity firms that happen to face lower market access costs and are able to export, while there could be high-productivity firms that face an unfavorable draw of high market access costs and are unable to export. Amand and Pelgrin (2016) show that when there is a wedge between exports data and productivity even a perfectly Pareto-distributed productivity can yield observable exports data that looks log-normal. However, with sufficiently large sample size exports data may provide a consistent estimator of a Pareto distribution for productivity growth when the wedge is not too heavy tailed and not too correlated with productivity at the top of the distribution (Amand and Pelgrin, 2016).

4.2 Structural Parameters of Firm Heterogeneity

We now turn to calculating the elasticity of substitution in each GTAP manufacturing sector such that we have a complete set of structural parameters of firm heterogeneity. Figure 7 shows the values of substitution elasticities across varieties in 19 GTAP manufacturing sectors.

There are four alternative calculations of elasticity values based on two firm size and two firm productivity metrics. First column, i.e. laborprod-ppprevenue, uses operating revenue as firm size and labor productivity as firm productivity. Second column, laborprod-employees, reports the elasticity values when number of employees and labor productivity are used for firm size and productivity, respectively. Third column, i.e. tfp-ppprevenue, reports the elasticity values when operating revenue and index TFP are used for firm size and productivity, respectively. Finally, fourth column, i.e. tfp-employees, reports the elasticity values when number of employees and index TFP are used for firm size and productivity, respectively.

There is a slight variation of elasticity values across 19 GTAP sectors, varying between the range of 1.9 to 2.8. Moreover, the distribution of elasticity values across sectors are quite similar when different metrics for firm size and productivity are used. The driving force for the distribution seems to be the firm size proxy. For example, Columns 1 and 3 are quite similar in both the distribution and value which both use operating revenue as firm size with different productivity metrics. Similarly, Columns 2 and 4 also have similar distributions and elasticity values, which use number of employees as firm size.

The narrow variation of elasticity values is not unexpected as there is narrow variation of power-law exponents in firm size and productivity as depicted in Figure 2. The range of values and narrow variation of elasticities are also encountered in studies such as Crozet and Koenig (2010) and Spearot (2016). For example, in Spearot (2016) average shape parameter values for productivity are estimated using trade and tariff data based on a Melitz and Ottaviano (2008) model using the GTAP database. He estimates the shape parameter for all GTAP sectors including 16 manufacturing sectors. The average shape values vary between 1.79-3.80 for manufacturing sectors. There is much higher variation across sectors when agricultural sectors are also taken into account.

4.3 Sensitivity Analysis

While we use optimized x_{min} values following Clauset, Shalizi, and Newman (2009), the corresponding revenue cutoff in each sector reduces the sample size considerably. Moreover, the size of the remaining sample differs across sectors which indicates that the power-law fit is conducted with the largest firms in some sectors, whereas, it is conducted with relatively smaller firms in others.

In order to observe the effect of constraining sample size on the power-law exponent, we re-estimated the power law in operating revenue for all firms in each sector while moving the lower firm size cutoff incrementally, i.e. x_{min}. Data is varied in uniform intervals. Plots are displayed between 60th and 99th percentile. The results are depicted in Figure 8 which plots α estimates against x_{min} values for 19 GTAP manufacturing sectors. As we move to the right on the horizontal axis, the power-law fit is conducted based on a smaller sample with larger firms. The optimal x_{min} value is indicated by a blue square, while the red line plots the α values based on the variation of x_{min} values in the range of 60th to 99th percentile of operating revenue.

Figure 8 shows that as we constrain the sample to larger firms, the power-law exponent increases in absolute value. We observe that in several sectors, there is a kink point of minimum firm size. The optimal x_{min} values mostly the tail of the distribution and are in a reasonable range of the plot. However, in sectors such as eeq, chm and ome, the optimal x_{min} values only use 1% of the tail distribution which cuts all the information out. This indicates that these sectors may not be following a power-law distribution. The performance of alternative distributions such as log-normal can be tested in such sectors.

5. Concluding remarks

In this paper, we estimate the structural parameters of firm heterogeneity in 19 GTAP manufacturing sectors, namely the shape parameter of productivity distribution and the elasticity of substitution across varieties using ORBIS firm-level data. We combine the methodology in Ahmad and Akgul (2018) and Clauset, Shalizi, and Newman (2009), where we first estimate the lower-bound for power-law minimizing the KS distance statistic between CDFs of the data and the model. We then estimate the power-law exponent of firm size and productivity based on Maximum Likelihood. The resulting power-law exponents are used to impute the elasticity of substitution. In addition, we compare the power-law fit to alternative distributions such as the exponential and log-normal distributions using likelihood ratio test. Finally we conduct sensitivity analysis using different lower-bounds.

We observe that there is a slight variation in the value of power-law exponents across GTAP sectors. Values are between 1.5 - 3.5. When firm size is measured by operating revenue, the distribution is slightly narrower with values between 1.5 - 2.6, while when firm size is measured by number of employees, the distribution is wider with values ranging between 1.5 - 3.4. This result is in line with power-law exponent estimates in the empirical literature.

There is a slight variation of elasticity values across 19 GTAP sectors, varying between the range of 1.9 to 2.8. Moreover, the distribution of elasticity values across sectors are quite similar when different metrics for firm size and productivity are used. The driving force for the distribution seems to be the firm size proxy. The narrow variation of elasticity values is is stemming from the narrow variation of power-law exponents in firm size and productivity.

Based on the LR tests and the CCDF fits, we observe that while labor productivity may follow a power-law model, operating revenue may not and look log-normal in some GTAP sectors. This indicates that there may be additional sources of heterogeneity besides productivity in some sectors such as heterogeneous market access costs or uncertainty that may be informing firm size and productivity.

It is also quite interesting to see that the empirical distributions considered in this study have concave down trends in log-log plots, therefore showing systematic deviations from power law behavior particularly at the upper tail of the fit interval. This suggests that applying a power-law fit for firm size and productivity in all GTAP sectors may not be accurate and not fully capture the distribution behavior.

References

Aguiar, A., M. Chepeliev, E. Corong, R. McDougall, and D. van der Mensbrugghe. 2019. "The GTAP Data Base: Version 10." *Journal of Global Economic Analysis*, 4(1).

Ahmad, S., and Z. Akgul. 2018. "Taking Melitz Seriously: A Simple Approach for Identifying Structural Parameters of Trade Models with Firm Heterogeneity." *USITC Working Paper*, pp. .

Alstott, J., E. Bullmore, and D. Plenz. 2014. "powerlaw: A Python Package for Analysis of Heavy-Tailed Distributions." *PLOS ONE*, 9(1).

Amand, M., and F. Pelgrin. 2016. "Pareto Distributions in International Trade: Hard to Identify, Easy to Estimate.", pp. . https://ssrn.com/abstract=2879623orhttp://dx.doi.org/10.2139/ssrn.2879623.

Axtell, R.L. 2001. "Zipf distribution of US firm sizes." *Science*, 293(5536): 1818–1820. doi:10.1126/science.1062081.

Balistreri, E.J., R.H. Hillberry, and T.F. Rutherford. 2011. "Structural estimation and solution of international trade models with heterogeneous firms." *Journal of International Economics*, 83(2): 95–108. doi:10.2139/ssrn.1153944.

Broda, C., and D.E. Weinstein. 2006. "Globalization and the Gains from Variety." *Quarterly Journal of Economics*, 121(2): 541–585. doi:10.1162/qjec.2006.121.2.541.

Brzezinski, M. 2014. "Do wealth distributions follow power laws? Evidence from rich lists." *Physica A: Statistical Mechanics and its Applications*, 406(155162).

Chaney, T. 2008. "Distorted Gravity: The Intensive and Extensive Margins of International Trade." *American Economic Review*, 98(4): 1707–1721. doi:10.1257/aer.98.4.1707.

Clauset, A., C.R. Shalizi, and M.E. Newman. 2009. "Power-law distributions in empirical data." *SIAM review*, 51(4): 661–703.

Crozet, M., and P. Koenig. 2010. "Structural gravity equations with intensive and extensive margins." *Canadian Journal of Economics-Revue Canadienne D Economique*, 43(1): 41–62. doi:10.1111/j.1540-5982.2009.01563.x.

Del Gatto, M., A. Di Liberto, and C. Petraglia. 2011. "Measuring productivity." *Journal of Economic Surveys*, 25(5): 952–1008.

di Giovanni, J., and A.A. Levchenko. 2013. "Firm entry, trade, and welfare in Zipf's world." *Journal of International Economics*, 89(2): 283–296. doi:10.3386/w16313.

di Giovanni, J., A.A. Levchenko, and R. Rancire. 2011. "Power laws in firm size and openness to trade: Measurement and implications." *Journal of International Economics*, 85(1): 42–52. doi:10.1016/j.jinteco.2011.05.003.

Eaton, J., S. Kortum, and F. Kramarz. 2011. "An Anatomy of International Trade: Evidence from French Firms." *Econometrica*, 79(5): 1453–1498. doi:10.3386/w14610.

Melitz, M.J. 2003. "The impact of trade on intra-industry reallocations and aggregate industry productivity." *Econometrica*, 71(6): 1695–1725. doi:10.3386/w8881.

Melitz, M.J., and S.J. Redding. 2013. "Firm Heterogeneity and Aggregate Welfare." *National Bureau of Economic Research*, , (18919)pp. .

Solow, R.M. 1957. "Technical change and the aggregate production function." *The review of Eco-*

nomics and Statistics, pp. 312–320.

Spearot, A. 2016. "Unpacking the Long Run Effects of Tariff Shocks: New Structural Implications from Firm Heterogeneity Models." *AEJ Microeconomics*, 8(2): 128–67. doi:10.1257/mic.20140015.

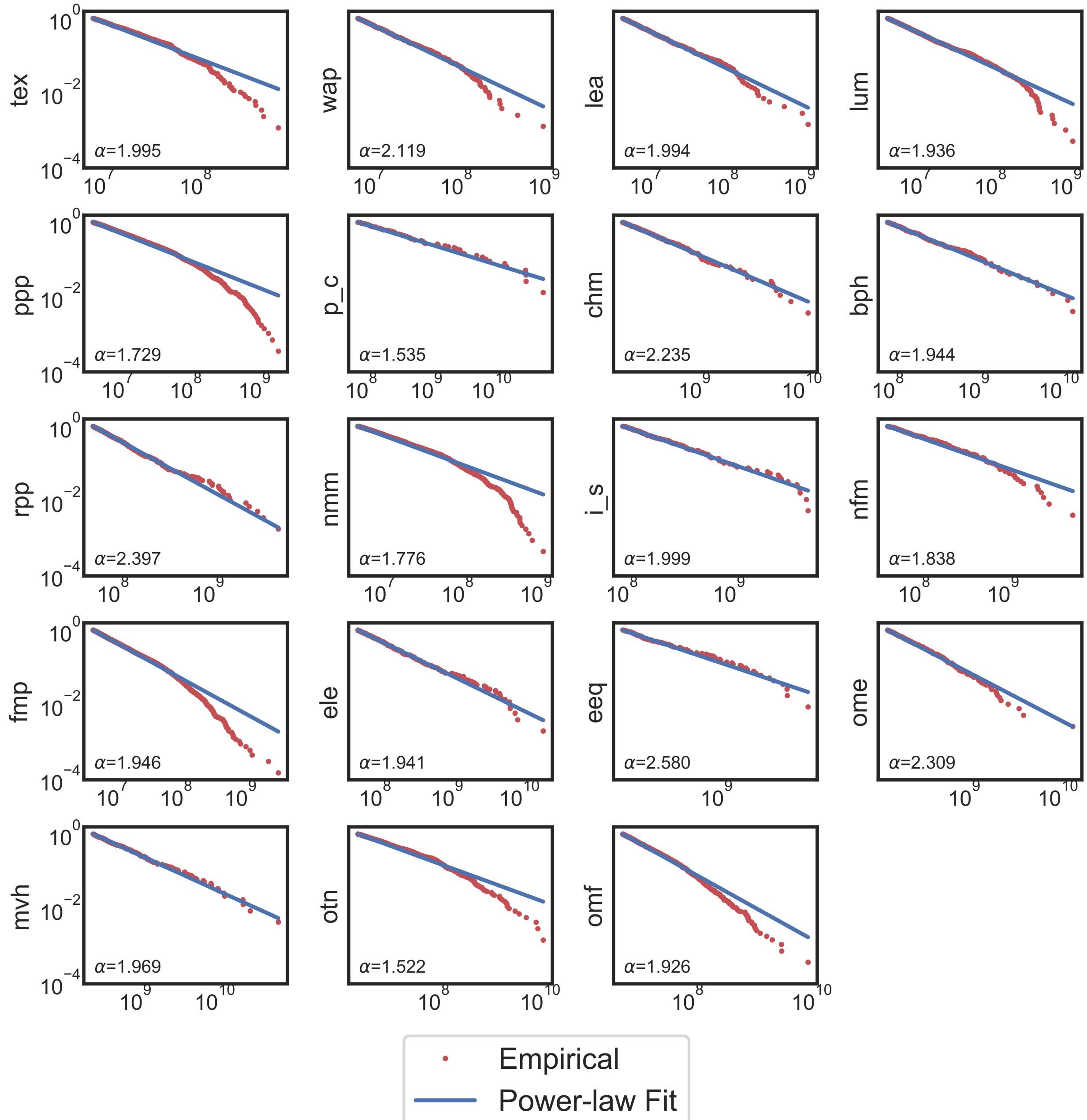

Figure 3. Empirical CCDF and Power Law Fit in Operating Revenue Distribution of GTAP Manufacturing Sectors (EU excluding Italy)

Notes: Complementary cumulative distribution function of the empirical firm size and productivity data and fitted power law distribution with a lower bound in the manufacturing sector of the EU excluding Italy - Operating Revenue.

Source: Author calculations.

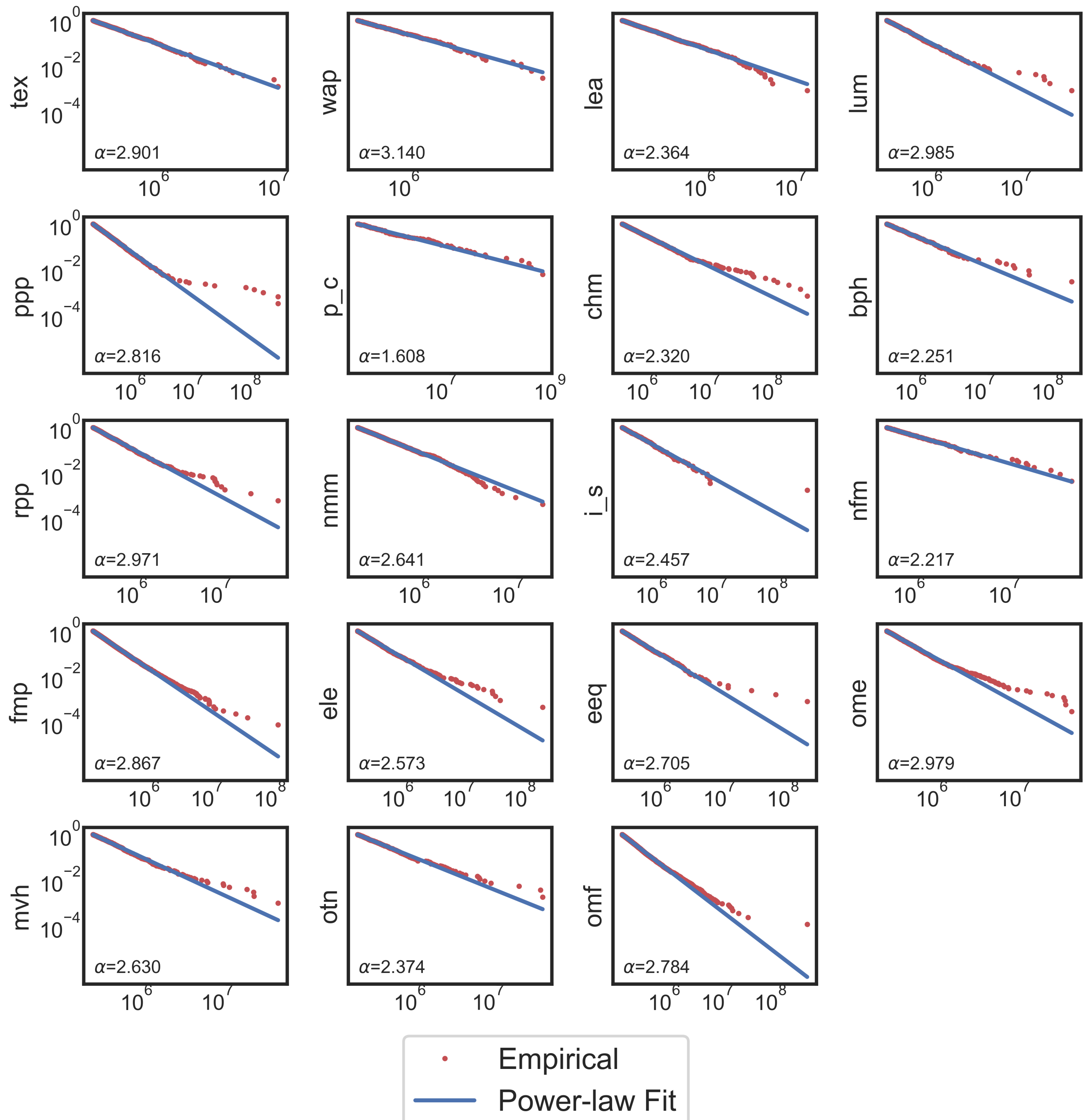

Figure 4. Empirical CCDF and Power Law Fit in Labor Productivity Distribution of GTAP Manufacturing Sectors (EU excluding Italy)

Notes: Complementary cumulative distribution function of the empirical firm size and productivity data and fitted power law distribution with a lower bound in the manufacturing sector of the EU excluding Italy - Labor Productivity.

Source: Author calculations.

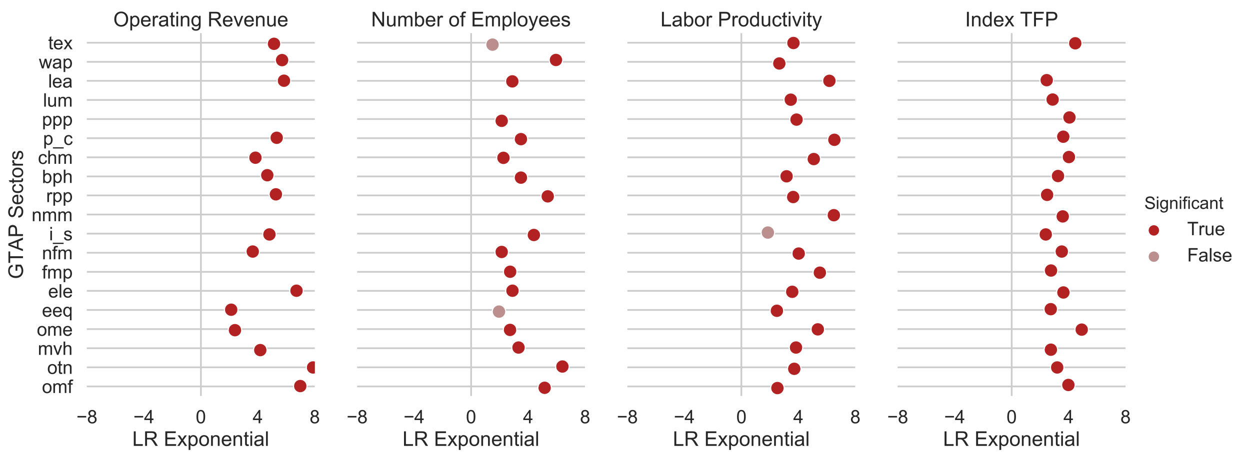

Figure 5. Likelihood Ratio Test against Exponential Distribution in GTAP Manufacturing Sectors (EU excluding Italy)

Notes: Likelihood Ratio Test against Exponential Distribution in GTAP manufacturing sectors of the EU excluding Italy (A) Operating Revenue, (B) Number of Employees, (C) Labor Productivity, and (D) Index Total Factor Productivity.

Source: Author calculations.

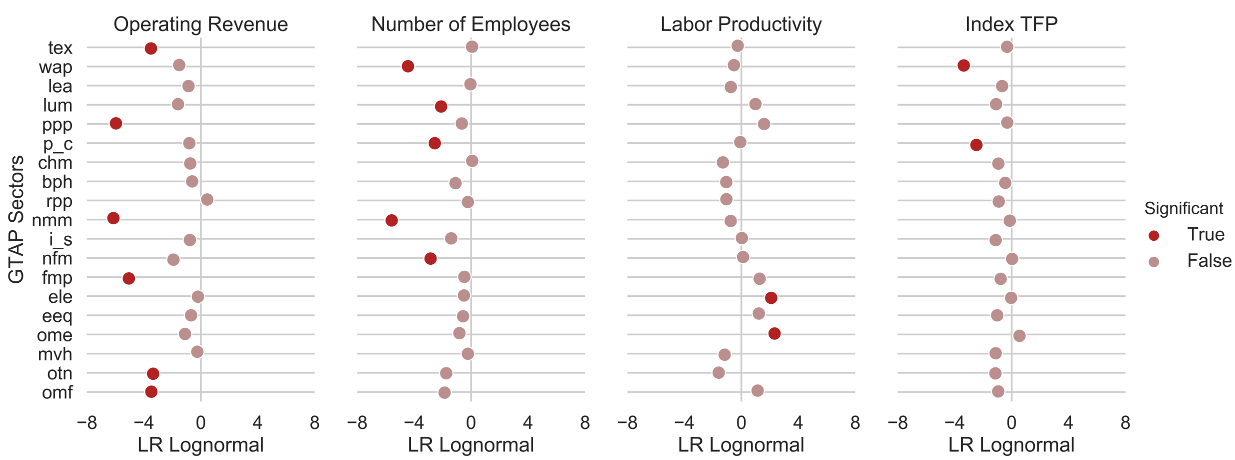

Figure 6. Likelihood Ratio Test against Log-normal Distribution in GTAP Manufacturing Sectors (EU excluding Italy)

Notes: Likelihood Ratio Test in GTAP manufacturing sectors of the EU excluding Italy (A) Operating Revenue, (B) Number of Employees, (C) Labor Productivity, and (D) Index Total Factor Productivity.

Source: Author calculations.

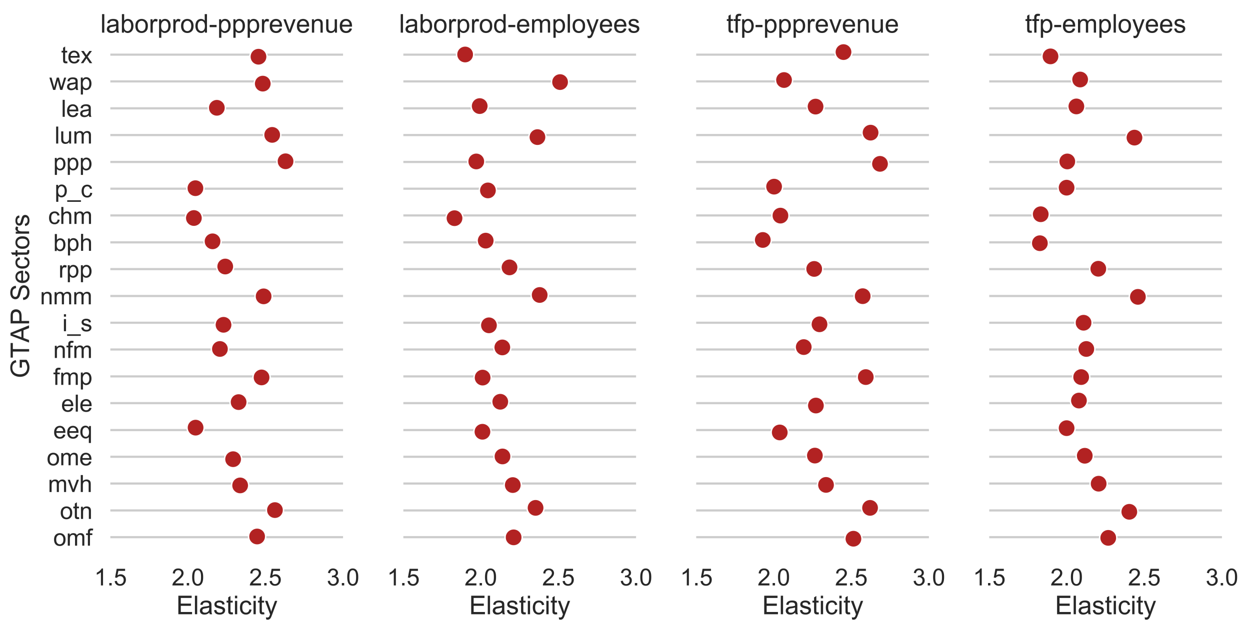

Figure 7. Elasticity Values in GTAP Manufacturing Sectors (EU excluding Italy)

Notes: Elasticity Values in GTAP manufacturing sectors of the EU excluding Italy based on four different measurements (A) Firm Size: Operating Revenue, Firm Productivity: Labor Productivity (B) Firm Size: Number of Employees, Firm Productivity: Labor Productivity (C) Firm Size: Operating Revenue, Firm Productivity: Index TFP, and (D) Firm Size: Number of Employees, Firm Productivity: Index TFP.

Source: Author calculations.

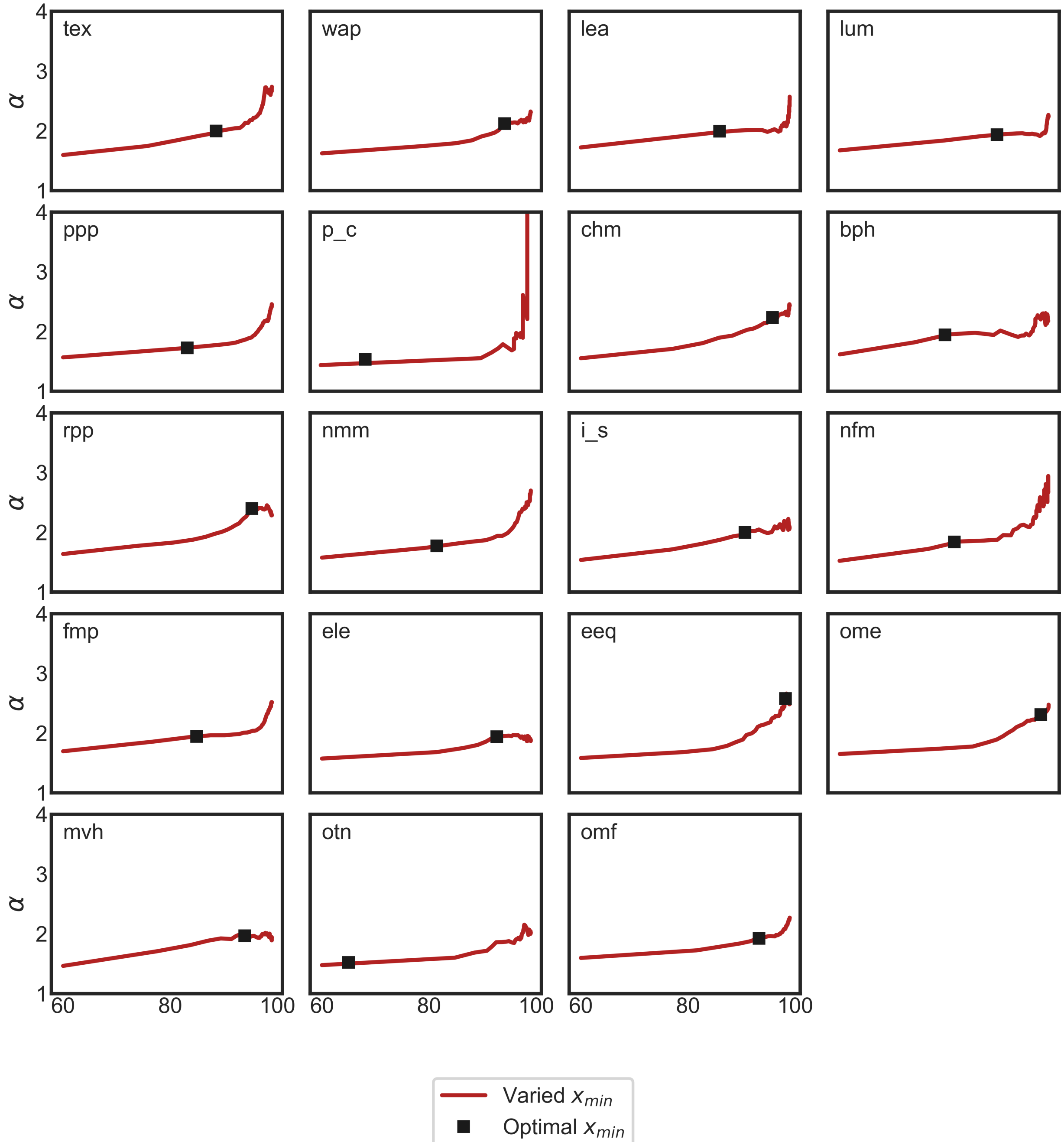

Figure 8. The Effect of Lower-Bound on Power-Law Exponent for Operating Revenue in GTAP Manufacturing Sectors (EU excluding Italy)

Notes: Power-law exponent against Minimum Firm Size in GTAP manufacturing sectors of the EU excluding Italy using Operating Revenue, (B) Number of Employees, (C) Labor Productivity, and (D) Index Total Factor Productivity.

Source: Author calculations.

Appendix A. Pareto Distribution

Following di Giovanni, Levchenko, and Rancire (2011), let's assume firm productivity has the Pareto CDF according to Equation (2) under autarky as follows

$$\Pr\left[\varphi \geq z\right] = \left(\frac{z}{x_{min}}\right)^{-\gamma} \tag{A.1}$$

where φ is the productivity of the firm and γ is the shape parameter of the productivity distribution, and x_{min} is the minimum level of productivity, which is the pareto scale parameter. The optimal demand and price for each variety in the firm heterogeneity model yields the following domestic sales by firm:

$$\begin{aligned}
S_i &= p_i q_i \\
&= \frac{Y}{P^{1-\sigma}} {p_i}^{1-\sigma} \\
&= \frac{Y}{P^{1-\sigma}} \left(\frac{\sigma}{\sigma-1} W\right)^{1-\sigma} {\varphi_i}^{\sigma-1} \\
&= A\varphi^{\sigma-1}
\end{aligned}$$

where p_i is the price charged by the heterogeneous firm i in the monopolistically competitive sector, q_i is the demand for firm i's variety, Y is the aggregate demand, P is the aggregate price index, W is the cost of factor payments, and $A = \frac{Y}{P^{1-\sigma}}(\frac{\sigma}{\sigma-1}W)^{1-\sigma}$. When φ follows the Pareto distribution, the power law of firm sales, $\S_i$ follows is given by:

$$\begin{aligned}
\Pr\left[S_i \geq s\right] &= \Pr\left[A\varphi^{\sigma-1} \geq s\right] \\
&= \Pr\left[\varphi^{\sigma-1} \geq \frac{s}{A}\right] \\
&= \Pr\left[\varphi \geq \left(\frac{s}{A}\right)^{\frac{1}{\sigma-1}}\right] \\
&= \left[\frac{1}{x_{min}}\left(\frac{s}{A}\right)^{\frac{1}{\sigma-1}}\right]^{-\gamma} \\
&= \left(x_{min}^{1-\sigma} A\right)^{\frac{\gamma}{\sigma-1}} s^{-\frac{\gamma}{\sigma-1}}
\end{aligned}$$

which satisfies Equation (2) for $X = S_i$, $x = s$, $C = \left(x_{min}^{1-\sigma} A\right)^{\frac{\gamma}{\sigma-1}}$, $A = \frac{Y}{P^{1-\sigma}}(\frac{\sigma}{\sigma-1}W)^{1-\sigma}$, and $\zeta = \frac{\gamma}{\sigma-1}$.

Appendix B. Complementary Cumulative Distribution Function

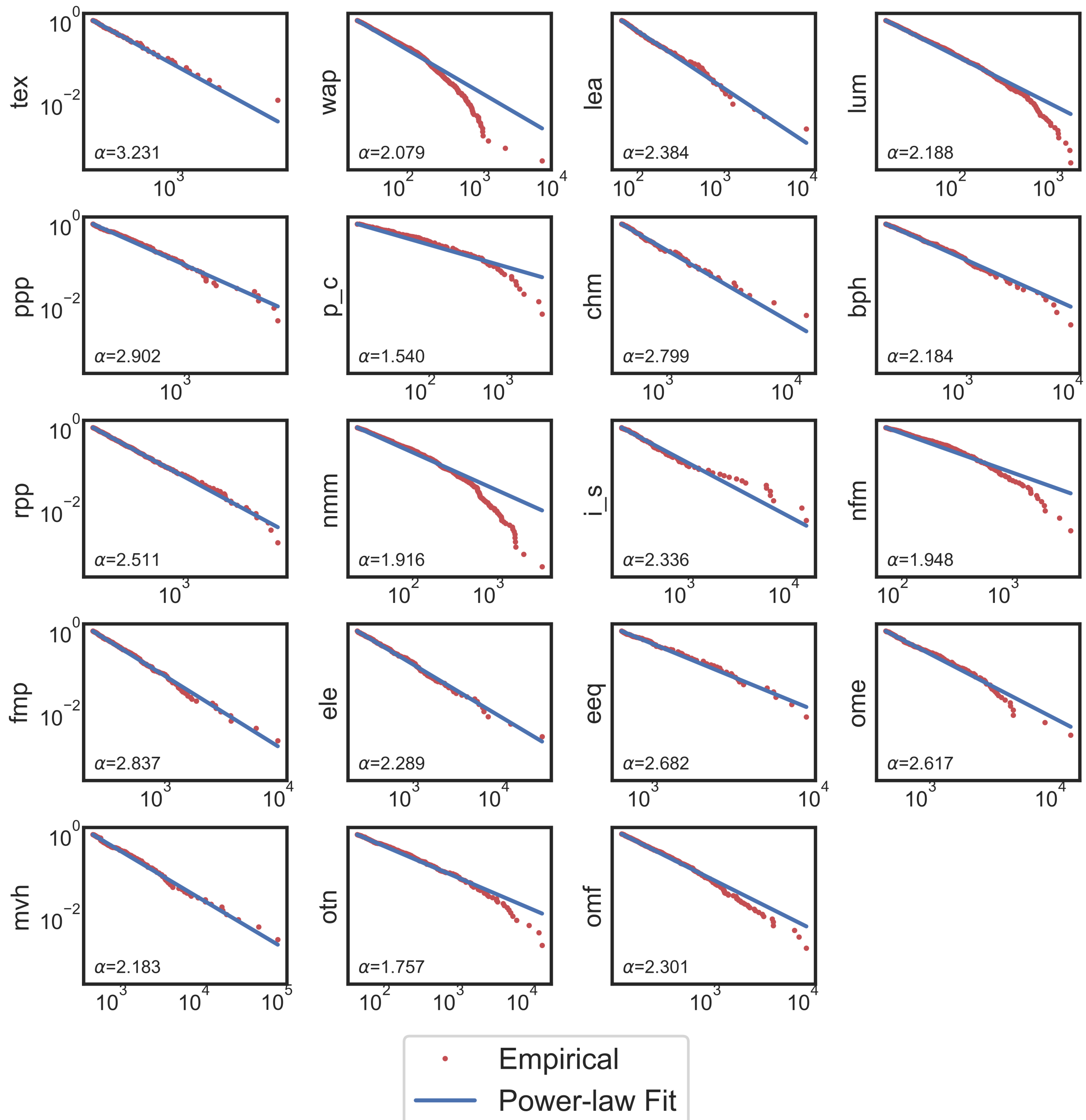

Figure B.1. Empirical CCDF and Power Law Fit in Number of Employees Distribution of GTAP Manufacturing Sectors (EU excluding Italy)

Notes: Complementary cumulative distribution function of the empirical firm size and productivity data and fitted power law distribution with a lower bound in the manufacturing sector of the EU excluding Italy - Number of Employees.

Source: Author calculations.

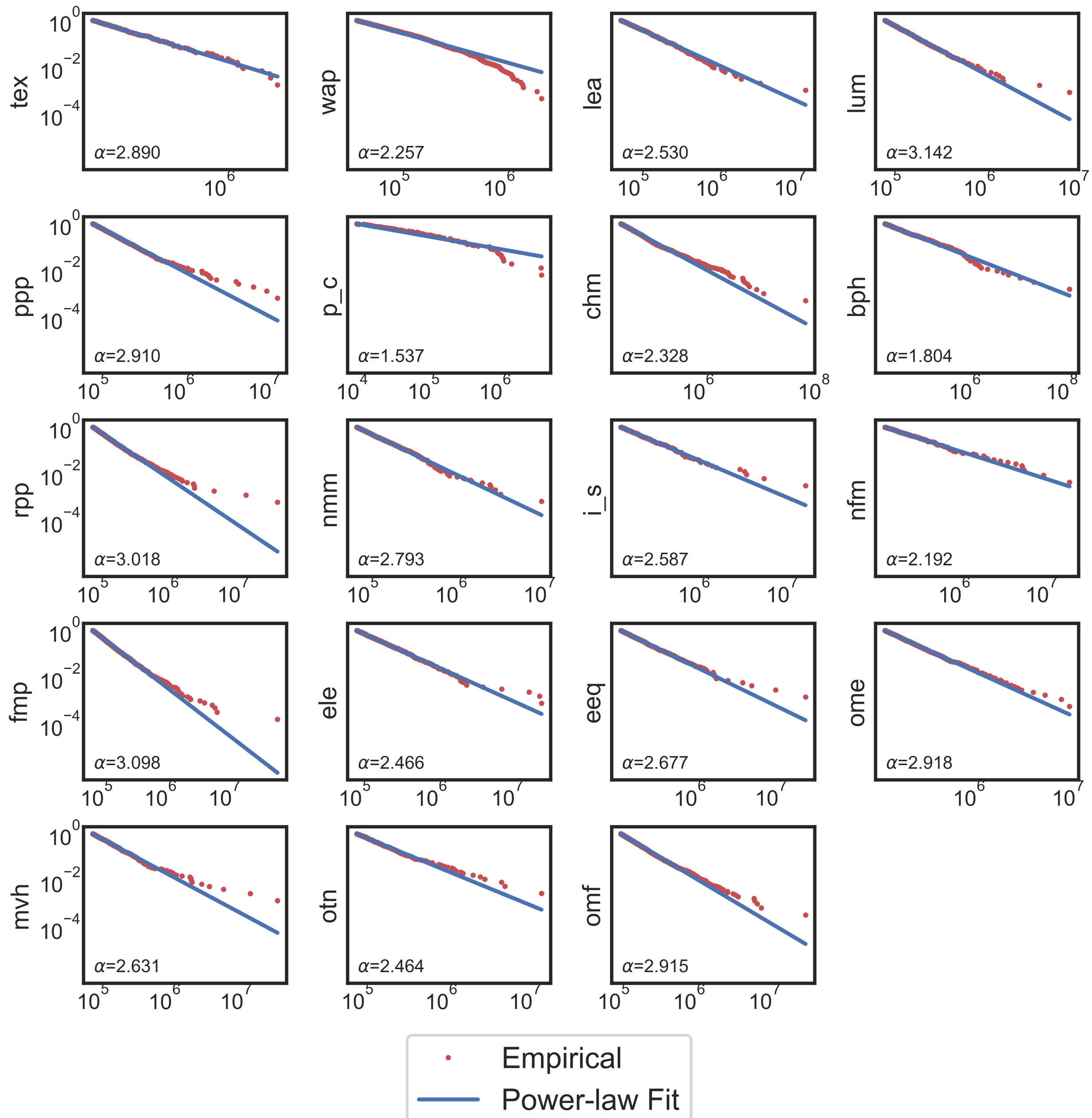

Figure B.2. Empirical CCDF and Power Law Fit in Index TFP Distribution of GTAP Manufacturing Sectors (EU excluding Italy)

Notes: Complementary cumulative distribution function of the empirical firm size and productivity data and fitted power law distribution with a lower bound in the manufacturing sector of the EU excluding Italy - Index Total Factor Productivity.

Source: Author calculations.